The Empire Open Math Contest

Question 1. Derek the Deli owner recently realized he is running low on ham. The total number of pounds of ham he has left is given by:

$$H = \frac{5}{4} + \sum_{n=2}^{\infty} \left[5^{\log_{(\sqrt[n]{n})}\left(\frac{1}{2^{2\log_5(n)}}\right)} \right] = \frac{5}{4} + 5^{\log_{\sqrt{2}}\left(\frac{1}{4^{\log_5(2)}}\right)} + 5^{\log_{\sqrt[3]{3}}\left(\frac{1}{4^{\log_5(3)}}\right)} + ...$$

A mysterious omniscient force tells us that this number H can be represented by a fraction of integers $\frac{p}{q}$ with p, q sharing no natural number factors in common other than 1. Compute $|p + q|$

Question 2. Billy the Boar just stumbled upon a magic number generator while riding the G train. The magic number generator accepts an integer x and produces an output $|x^3 + 9x^2 + 15x + 7|$. For how many distinct integers x does Billy's magic number generator produce a prime number?

Question 3. The year is 1776... John Adams rolls once a fair 20-sided Icosahedral die labelled (0-19) on each of its sides. Immediately after this Benjamin Franklin then rolls once a fair 12-sided dodecahedral die labelled (0-11) on each side. Finally John Hancock rolls once a fair 8-sided octahedral die labelled (0-7) on each side. Let P be the probability that the product of all the 3 rolls is divisible by 7.

Another mysterious omniscient force tells us that this number P can be represented by a fraction of natural numbers $\frac{p}{q}$ with p, q sharing no factors in common other than 1. Compute $p + q$

Question 4. (BONUS) Tommy the Tiger was buying some fancy jewelry in Jackson heights when he suddenly noticed a mysterious 3000 carrot diamond cube behind one of the counters. The Jeweler offered him the following discount plan

1) A 25% discount if he could compute
   - first the number of colorings of the faces of a cube using 6 colors (each a single time) in a fixed orientation (without considering any symmetries such as rotation/reflection). Call this A.
   - secondly the number of colorings of the points of a cube in a fixed orientation (without considering any symmetries such as rotation/reflection) using the colors red and blue. Call this B

   Compute the value $|A - B|$ and write it down in problem 4 of the answer sheet.
2) A 100% discount if he could compute
   - firstly number of distinct colorings of the faces of a cube using 2 distinct colors while considering rotational symmetry, call this A.
   - secondly the number of distinct colorings of the edges of a cube using 2 distinct colors while considering rotational symmetry, call this B
   - thirdly the number of distinct colorings of the vertices of a cube using 2 distinct colors while considering rotational symmetry, call this C

   Compute $|A - B + C|$ and write it down in problem 11 of the answer sheet.

Solving the first section (problem 4) counts as 1 point. Solving the second harder section (problem 11) will count for 2 additional points.

Question 5. The Egyptian God of the Underworld Anubis has decided to offer you a special hilton-gold-honors-discount-redeem-anytime-fast-pass to the after life in place of the usual spiritual trials if you can solve this 2022 version of a problem he saw in the 2002 BC AMC 12.

Let

$$a_n = \begin{Bmatrix} 2, & \text{if } n \text{ is divisible by 3 and 337} \\ 3, & \text{if } n \text{ is divisible by 2 and 337} \\ 337, & \text{if } n \text{ is divisible by 2 and 3} \\ 0, & \text{otherwise} \end{Bmatrix}$$

Compute the value of

$$\sum_{n=1}^{2021} a_n$$

Question 6. (Originally Proposed by Terence Coelho, Ph.D Rutgers University) Nicholas Cage invited you to his birthday party in 4-dimensional Euclidean Space $\mathbb{R}^4$ with origin O. While at this birthday party you and the guests begin to a play a variant of the game of twister where all of you are compressed into a single point in the origin O except for your limbs which are transformed into rays from the origin. Every guest has 2 hands so each guest transforms into exactly 2 rays coming from the origin. A valid configuration of 4 dimensional twister is a set of rays emerging from the origin such that every pair of rays in this set has an obtuse angle between them. What is the maximal number of guests that can play this game before it becomes impossible to find a valid configuration of 4 dimensional twister.

Question 7. Samantha the Software Engineer was counting her stacks of cash this past Sunday. She was part of an unusual stock vesting plan where she is granted $n^4 + 2n^2 + 1$ shares per day for her n^{th} day of working (with her first day being $n = 1$). We wish to compute how many shares she has earned on her 100th day of working call this number B. For this answer just to compute the last 4 digits of B.

Question 8. You decide to visit Starbucks$^{\text{tm}}$ one day and order an unusually large cup of coffee. When you receive your cup you notice there is some unusual coffee art on the cup. The art can be described as a brown "coffee region" on a white background. We describe the coffee region below:

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 8x - 2$
- Now consider the region of $\mathbb{R}^2$ given by the intersection of $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. We shall call this the brown "coffee region".

Compute the value of the area of this region. Round your answer to the nearest integer. If the answer falls exactly at an integer $p + 0.5$, then round the answer down.

Question 9. Gandalf, Galadriel and the squad were fighting Sauron in Dol Gudur. After a few tough hits from Sauron, Gandalf and Galadriel decide to unleash a very powerful spell which consists of drawing a regular heptagon inside a unit circle and connecting every pair of vertices of this heptagon with a line segment. Find the sum of the squares of all these line segments. Round your answer to the nearest integer. If the answer falls exactly at an integer $p + 0.5$, then round the answer down.

Question 10. (Originally proposed by Terence Coelho, Ph.D Rutgers University) In the exotic foreign planet known as Philadelphia, PA, A day consists of 25 hours, an hour consists of 61 minutes, a minute consists of 61 seconds and a second consists of 1001 miniseconds (every second the minisecond hand makes a full revolution). Suppose you have a Philadelphia clock which has an hour hand, minute hand, second hand and minisecond hand that each move at a constant speed throughout the day according to the above rates. The clock has 25 hour labels all evenly spaced, so unlike a regular clock, the hour hand makes ONE revolution of this clock per day. When the day starts, all the hands are aligned facing up. Over the course of a day how many times will all 4 hands perfectly line up?