The Empire Open Math Contest

Question 1. Derek the Deli owner recently realized he is running low on ham. The total number of pounds of ham he has left is given by:

$$H = \frac{5}{4} + \sum_{n=2}^{\infty} \left[5^{\log_{(\sqrt[n]{n})}\left(\frac{1}{2^{2\log_5(n)}}\right)} \right] = \frac{5}{4} + 5^{\log_{\sqrt{2}}\left(\frac{1}{4^{\log_5(2)}}\right)} + 5^{\log_{\sqrt[3]{3}}\left(\frac{1}{4^{\log_5(3)}}\right)} + \dots.$$

A mysterious omniscient force tells us that this number H can be represented by a fraction of integers $\frac{p}{q}$ with p, q sharing no natural number factors in common other than 1. Compute $|p+q|$

Solution: So there is one less than obvious logarithm identity that needs to be noticed:

$$\log_{n^a}(x) = \frac{1}{a}\log_n(x)$$

Playing with $a = \frac{1}{2}, n = 2$ makes it obvious why this is true. $\sqrt{2}^{2n} = 2^n$.

$$5^{\log_{(\sqrt[n]{n})}\left(\frac{1}{2^{2\log_5(n)}}\right)} = 5^{\log_{n^{\frac{1}{n}}}\left(\frac{1}{2^{2\log_5(n)}}\right)} = 5^{n\log_n\left(\frac{1}{2^{2\log_5(n)}}\right)} = 5^{2\log_5(n) n \log_n} = \left(\frac{1}{2}\right) n^{2\log_n(\frac{1}{2})} = \left(\frac{1}{2}\right)^{2n}$$

Now we observe that $\frac{5}{4} = 1 + \left(\frac{1}{2}\right)^2$ So really this entire problem is just computing

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n}$$

Using the geometric series formula we calculate this to be $\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ So our final value is $3 + 4 = 7$.

Question 2. Billy the Boar just stumbled upon a magic number generator while riding the G train. The magic number generator accepts an integer x and produces an output $|x^3 + 9x^2 + 15x + 7|$. For how many distinct integers x does Billy's magic number generator produce a prime number?

Solution: So for this one we need to factor $x^3 + 9x^2 + 15x + 7$. Newtons formulas tell us that if this factors we could factor this into $(x+r_1)(x+r_2)(x+r_3) = x^3 + (r_1+r_2+r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x + r_1r_2r_3$ if we know what the magic r_1, r_2, r_3 are. Staring at the equations

$$r_1 + r_2 + r_3 = 9$$
$$r_1r_2r_3 = 7$$

It becomes apparent

$$1 + 1 + 7 = 9$$
$$1 * 1 * 7 = 7$$

So we factor this into $(x+1)^2(x+7)$. Recall a prime number is only divisible by the factors 1 and itself. Since $(x+1)^2$ is a repeated factor we need that $(x+1)^2 = 1$. This means that either $x = 0, x = -2$. Sure enough checking both we see that $(0+7) = 7$ and $(-2+7) = 5$ so both are prime and we have an answer of 2.

Question 3. The year is 1776... John Adams rolls once a fair 20-sided Icosahedral die labelled (0-19) on each of its sides. Immediately after this Benjamin Franklin then rolls once a fair 12-sided dodecahedral die labelled (0-11) on each side. Finally John Hancock rolls once a fair 8-sided octahedral die labelled (0-7) on each side. Let P be the probability that the product of all the 3 rolls is divisible by 7.

Another mysterious omniscient force tells us that this number P can be represented by a fraction of natural numbers $\frac{p}{q}$ with p, q sharing no factors in common other than 1. Compute $p + q$

Solution: (Credit: Terence Coelho) We compute the probability that the product of the rolls is NOT divisible by 7 (call it A) and compute $1 - A$ to get the probability that the product is divisible 7. So the product is not divisible by 7 if none of the dice rolled something divisible by 7. There are $20-3$ ways to do this with the first die (dont forget the 0!), $12-2$ ways to do this with the second die and $8-2$ ways to do this with the last die. So the probability we want to compute is

$$1-\frac{17}{20}\frac{10}{12}\frac{6}{8}$$

Making some simplifications this is:

$$1-\frac{17}{20}\frac{5}{6}\frac{3}{4}=1-\frac{17}{20}\frac{5}{2}\frac{1}{4}=1-\frac{17}{4*2*4}=1-\frac{17}{32}=\frac{15}{32}$$

Since these are in lowest terms we compute $15+32=47$.

Question 4. (BONUS) Tommy the Tiger was buying some fancy jewelry in Jackson heights when he suddenly noticed a mysterious 3000 carrot diamond cube behind one of the counters. The Jeweler offered him the following discount plan

1) A 25% discount if he could compute
   - first the number of colorings of the faces of a cube using 6 colors (each a single time) in a fixed orientation (without considering any symmetries such as rotation/reflection). Call this A.
   - secondly the number of colorings of the points of a cube in a fixed orientation (without considering any symmetries such as rotation/reflection) using the colors red and blue. Call this B

   Compute the value $|A-B|$ and write it down in problem 4 of the answer sheet.
2) A 100% discount if he could compute
   - firstly number of distinct colorings of the faces of a cube using 2 distinct colors while considering rotational symmetry, call this A.
   - secondly the number of distinct colorings of the edges of a cube using 2 distinct colors while considering rotational symmetry, call this B
   - thirdly the number of distinct colorings of the vertices of a cube using 2 distinct colors while considering rotational symmetry, call this C

   Compute $|A-B+C|$ and write it down in problem 11 of the answer sheet.

Solving the first section (problem 4) counts as 1 point. Solving the second harder section (problem 11) will count for 2 additional points.

Solution Part 1: There are $A=6!=720$ ways to color the cube with 6 distinct colors each used once if we don't consider different rotations the same. (Just put the cube in a fixed orientation and labelled the sides 1-6. There are 6 choices for the color of side 1, there are 5 colors for the color of side 2, etc...). Now there are 8 vertices of a cube, so again we can label the vertices $1-8$, and now we color them as 2 choices for vertex 1, 2 choices for vertex 2, etc... giving us $B=2^8$ colorings in total.

$$|A-B|=6!-2^8=720-256=464$$

Solution Part 2: This problem definitely brutal if you don't have a physical cube in your hand to look at while solving it, and haven't learned group theory. The most direct way to solve this will require using Burnsides' lemma from group theory. This basically states that the number of orbits under a group action is equal to the sum of the count of the number of objects fixed by each element of the group all divided by the size of the group. Stated formally that is:

$$|X/G|=\frac{1}{|G|}\sum_{g\in G}|X^g|$$

So the use of this theorem arises from the fact that if we consider a configuration of the colored cube up to symmetry then we are considering an "orbit" of that specific colored cube under action of the symmetry group of the cube. So we really do want to count these orbits. This link provides a visual on all the symmtries: https://garsia.math.yorku.ca/ zabrocki/math4160w03/cubesyms/

we first classify all the rotational symmetries of the cube.

There is 1 identity symmetry which changes nothing about the cube (you just get the same cube you started with).

There are 6 distinct rotational symmetries of turning a particular face clockwise 90 degrees (It turns out that turning a face counter clockwise, is the same turning its opposite face clockwise so we have covered all the 90 degree turns this way). These symmetries cause 4 faces to move and and leave 2 faces fixed.

There are 3 rotational symmetries of the cube that involve turning about a face for 180 degrees (turning 180 degrees clockwise and counter clockwise are the same). These symmetries cause 4 faces to move (by swapping sides) and 2 faces to remain fixed.

There are 6 symmetries of the cube that involve rotation along the center of any edge by 180 degrees. This leaves the 2 edges fixed but moves all the other edges.

There are 8 symmetries of the cube that involve rotation around corner either 120 or 240 degrees. Since there are 4 such diagonals and 2 valid degrees of rotation we get 8 such symmetries. These fix 2 vertices and move everything else.

So the size of this symmetry group is $|G| = 1+6+3+6+8 = 24$ Now we count the number of red-blue face colorings up to symmetry. There are naively 2^6 such colorings which remain totally unchanged under the identity (the identity changes nothing so this is not a surprise). Under a 90 degree face rotation the only way a colored cube remains unchanged is if the middle 4 faces are all the same color. So there are 2^3 ways to color the cube and satisfy this property (1 color for the front face, 1 for the back face and 1 for the middle 4). Under a 180 degree face rotation we don't need the middle 4 to all be the same, just opposite side faces of the middle 4 need to be the same. So there are now 2^4 ways to color a cube that's invariant under this rotation. Now we consider rotation along the edges. This is hard to visualize so for ease its best to hold a cube in your hand. There will be 3 pairs of faces that swap sides so there are 2^3 ways to color the faces of a cube invariant under this. Lastly we have our corner rotations. This splits the faces of the cube into two groups of 3 each group of 3 needing to have the same color so there are only 2^2 way to do this. All in all coloring the faces of the cube we have

$$\frac{2^6 + 6 * 2^3 + 3 * 2^4 + 6 * 2^3 + 8 * 2^2}{24} = 10$$

Ways to do this. We are only 1/3 of the way done. We still have to color edges and vertices. We now try to do the edges. There are 2^{12} ways to color the edges of a cube invariant under the identity. There are 2^3 ways to color the edges of a cube invariant after a 90 degree face rotation. There are 2^6 ways to color the edges of a cube invariant after a 180 degree face rotation. There are 2^7 ways to color the edges of the cube with an edge rotation (the 2 opposite edges are each a degree of freedom and the remaining 10 edges get swapped with each other so only 5 additional degrees of freedom). Finally for the corner rotation there are 3 edges from each corner that are accounted for. The remaining 6 edges each form an orbit of size 3. So we have 4 degrees of freedom here. Thus we have

$$\frac{2^12 + 6 * 2^3 + 3 * 2^6 + 6 * 2^7 + 8 * 2^4}{24} = 218$$

Ways to uniquely color the edges. Now we do count coloring of the vertices. There are 2^8 colorings invariant under the identity. There are 2^2 colorings which are invariant under face rotation by 90 degrees. There are 2^4 colorings that are invariant under face rotation by 180 degrees. There are again 2^4 colors that are invariant under edge rotation by 180 degrees. Finally the point rotation analysis becomes easy there are 2^4 colorings invariant under point rotation (the two corners are free to color and the 6 middle vertices can be broken up into 2 groups of 3 vertices that each form their own independent orbit). Thus we compute

$$\frac{2^8 + 6 * 2^2 + 3 * 2^4 + 6 * 2^4 + 8 * 2^4}{24} = 23$$

Thus we compute $|10 - 218 + 23| = 185$

Question 5. The Egyptian God of the Underworld Anubis has decided to offer you a special hilton-gold-honors-discount-redeem-anytime-fast-pass to the after life in place of the usual spiritual trials if you can solve this 2022 version of a problem he saw in the 2002 BC AMC 12.

Let

$$a_n = \left\{ \begin{array}{cc} 2, & \text{if } n \text{ is divisible by 3 and 337} \\ 3, & \text{if } n \text{ is divisible by 2 and 337} \\ 337, & \text{if } n \text{ is divisible by 2 and 3} \\ 0, & \text{otherwise} \end{array} \right\}$$

Compute the value of

$$\sum_{n=1}^{2021} a_n$$

Solution: So for this problem we first observe that $2 * 3 * 337 = 2022$. So this function a_n only has its undefined value at 2022 but luckily for us that is not part of the sum. Our next important detail is to then count how many times will a_n take on a particular value.

So from 1...2022 inclusive there are only 2 values that are divisible by $3, 337$. Since we aren't considering 2022 in the sum then there is only 1 value. Through similar arguments: From 1...2022 inclusive there are only 3 values that are divisible by $2, 337$ but only 2 of those are not equal to 2022. Finally from 1...2022 there are 337 values divisible by $2, 3$ but only 336 values when not consider 2022. Thus our sum is equal to

$$2 * (1) + 3 * (2) + 337 * (336) = 8 + 337 * 336$$

Now this is some tedious arithmetic giving us $8 + 113232 = 113240$.

Question 6. (Originally Proposed by Terence Coelho, Ph.D Rutgers University) Nicholas Cage invited you to his birthday party in 4-dimensional Euclidean Space $\mathbb{R}^4$ with origin O. While at this birthday party you and the guests begin to a play a variant of the game of twister where all of you are compressed into a single point in the origin O except for your limbs which are transformed into rays from the origin. Every guest has 2 hands so each guest transforms into exactly 2 rays coming from the origin. A valid configuration of 4 dimensional twister is a set of rays emerging from the origin such that every pair of rays in this set has an obtuse angle between them. What is the maximal number of guests that can play this game before it becomes impossible to find a valid configuration of 4 dimensional twister.

Solution: So the observation that needs to be found here is that there can be at most $d + 1$ pairwise obtuse rays in R^d. We can prove this by induction: suppose its true for $1...d - 1$. Then in R^d you can consider a set of $d + 2$ rays. Lets fix one ray r. Then we definitely know that there isn't a ray directly opposite of r and we also can consider the hyperplane H_r which is normal to r and passing through the origin. Clearly all the other rays need to be on the other side of this hyperplane (rays in the hyperplane would have a 90 degree angle from r and so thats not possible). Now wlog we can pretend r is the unit vector e_0 so that the projection of the remaining the rays onto the hyperplane H_r amounts simply to taking those rays coordinates $(v_0, v_1, ...)$ and consider $(0, v_1, ...)$. So what does it mean for to 2 rays u^1, u^2 to be obtuse? it means the dot product between the rays is negative i.e. $\sum_{n=0}^{d} u_n^1 * u_n^2 < 0$. Now if we take the project into the hyperplane and compute the dot product the new dot product will be $\sum_{n=1}^{d} u_n^1 * u_n^2 = \sum_{n=0}^{d} u_n^1 * u_n^2 - u_0^1 u_0^2$. Now that expression $u_0^1 u_0^2$ must be positive because it is the product of 2 negative numbers (since they lie on the opposite side of the hyperplane through the origin normal to r) and so we conclude that $\sum_{n=1}^{d} u_n^1 * u_n^2 < \sum_{n=0}^{d} u_n^1 * u_n^2$. So our $d + 1$ rays when projected to the hyperplane must remain pairwise obtuse. But this is a contradiction since as part of our induction step we assumed that there cannot be more than d pairwise obtuse rays in R^{d-1} (and that hyperplane is clearly isomorphic to R^{d-1}). So we conclude the original $d + 1$ pairwise obtuse rays couldn't have existed, i.e. $d + 2$ pairwise obtuse rays cannot exists in R^d. Now its worth pointing out that the equilateral $d - simplex$ centered at the origin gives us a model of $d + 1$ pairwise obtuse rays (by considering the rays passing through each of the $d + 1$ corners). So we conclude that in R^d there can be at most $d + 1$ pairwise obtuse rays.

Now the problem has some fluff where every guest gets reduced to 2 rays and the party takes place in R^4 so we can have at most 5 rays, so at most 2 guests can play (a third guest would be too many).

Question 7. Samantha the Software Engineer was counting her stacks of cash this past Sunday. She was part of an unusual stock vesting plan where she is granted $n^4 + 2n^2 + 1$ shares per day for her n^{th} day of working (with her first day being $n = 1$). We wish to compute how many shares she has earned on her 100th day of working call this number B. For this answer just to compute the last 4 digits of B.

Solution: So there is a solution to this if you happen to know Faulhaber's Formula. We will assume you don't because honestly who wants to keep track of all those crazy bernoulli numbers. So we need to evaluate

$$\sum_{n=1}^{100} n^4 + 2n^2 + 1 \mod 1000$$

We can take a well informed guess that there is a family of polynomials $P(n)$ such that $P(n) - P(n-1) = n^4 + 2n^2 + 1$. It then follows that if $P(0) = 0$ then $P(1) = 1^4 + 2n^2 + 1$ and $P(2) = (2^4 + 2*2^2 + 1) + (1^4 + 2*1^2 + 1)$ etc... so $P(n) = \sum_{k=1}^{n} k^4 + 2k^2 + 1$. And so $P(100) \mod 1000$ is the quantity we are looking for. So now we get to work. We assume $P(n) = an^5 + bn^4 + cn^3 + dn^2 + en + f$ where the last constant f we will set to 0. Then the condition:

$$P(n) - P(n-1) = n^4 + 2n^2 + 1$$

Gives us:

$$a(n^5 - (n-1)^5) + b(n^4 - (n-1)^4) + c(n^3 - (n-1)^3) + d(n^2 - (n-1)^2) + e = n^4 + 2n^2 + 1$$

So now we need to expand these expressions via the binomial theorem

$$a(5n^4 - 10n^3 + 10n^2 - 5n + 1) + b(4n^3 - 6n^2 + 4n - 1) + c(3n^2 - 3n + 1) + d(2n - 1) + e = n^4 + 2n^2 + 1$$

So we find from this:

$$\begin{gathered} 5a = 1 \\ -10a + 4b = 0 \\ 10a - 6b + 3c = 2 \\ -5a + 4b - 3c + 2d = 0 \\ a - b + c - d + e = 1 \end{gathered}$$

Solving the equations from top to bottom we get $a = \frac{1}{5}, b = \frac{1}{2}, c = 1, d = 1, e = 1 - 1/5 + 1/2 - 1 + 1 = \frac{13}{10}$ So then we conclude that

$$P(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + n^3 + n^2 + \frac{13}{10}n$$

So now we need to evaluate $P(100) \mod 1000$ i.e.

$$\frac{1}{5}(100)^5 + \frac{1}{2}(100)^4 + (100)^3 + (100)^2 + \frac{13}{10}(100) \mod 1000$$

Its easy to check that all but the linear term are divisible by 1000 and so we are simply interested in

$$\frac{13}{10}(100) = 130$$

And that is the answer!

Question 8. You decide to visit Starbucks$^{\text{tm}}$ one day and order an unusually large cup of coffee. When you receive your cup you notice there is some unusual coffee art on the cup. The art can be described as a brown "coffee region" on a white background. We describe the coffee region below:

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 8x - 2$
- Now consider the region of $\mathbb{R}^2$ given by the intersection of $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. We shall call this the brown "coffee region".

Compute the value of the area of this region. Round your answer to the nearest integer. If the answer falls exactly at an integer $p + 0.5$, then round the answer down.

Solution: So we consider $f(x) + f(y) \leq 0 \to x^2 + 8x - 2 + y^2 + 8y - 2 \leq 0$. This can be simplified to $x^2 + 8x + y^2 + 8y \leq 4$. We can add 32 to both sides to yield $x^2 + 8x + 16 + y^2 + 8y + 16 \leq 36$. Which now can be factored into $(x + 4)^2 + (y + 4)^2 \leq 6^2$ Which we can recognize is a circle centered at $x = -4, y = -4$ with radius 6. Which has area 36π. Now we consider our next constraint $f(x) - f(y) \leq 0$. This is $x^2 + 8x + y^2 + 8y \leq 0$. This can regrouped as $x^2 - y^2 + 8x - 8y \leq 0$ and then factored into $(x - y)(x + y + 8) \leq 0$. This describes a cone with center at $x = -4, y = -4$. It's worth noting that $x - y = 0$ and $x + y + 8 = 0$ are perpendicular from each other (you can use the slope formula to check this) and so the cone splits the circle into exactly 4 equal sized sectors and only 2 of the sectors are be filled in by the cone inequality. So the total area is actually half of this or 18π. We now compute $18 * 3.14$ and round that answer.

Question 9. Gandalf, Galadriel and the squad were fighting Sauron in Dol Gudur. After a few tough hits from Sauron, Gandalf and Galadriel decide to unleash a very powerful spell which consists of drawing a regular heptagon inside a unit circle and connecting every pair of vertices of this heptagon with a line segment. Find the sum of the squares of all these line segments. Round your answer to the nearest integer. If the answer falls exactly at an integer $p + 0.5$, then round the answer down.

Solution: (By Terence Coelho) It helps to sketch a heptagon with all its sides connected inscribed in a circle. We do so below:

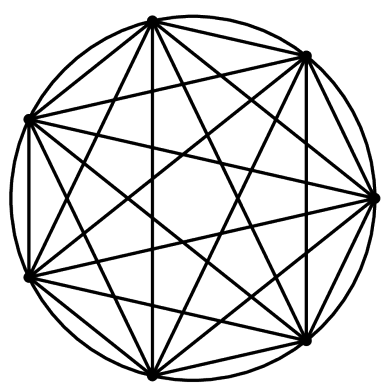

FIGURE 1. The graph

So these line segments seem intimidating at first until you recall that they are actually chords of a circle. And that means it's possible to understand the length of these chords using some simple formulae. We first categorize the chords. There are 7 chords between two adjacent vertices and so they are the chord of the angle in radians $\frac{2\pi}{7}$. There are there are 7 chords of the angle $\frac{4\pi}{7}$ in radians and lastly there are 7 chords of the angle $\frac{8\pi}{7}$ in radians. Now consider the diagram below:

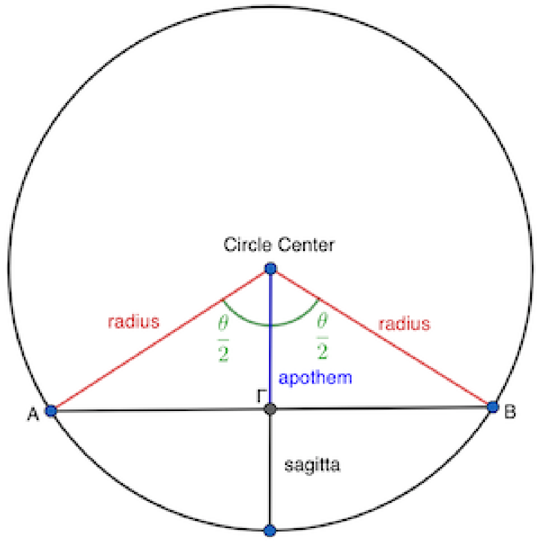

FIGURE 2. The chord length

It's clear that the length of the chord forms a triangle with 2 sides equal to radius of our circle. Since we are using the unit circle this radius is 1. So the length of this chord can be computed by the law of cosines $a^2 + b^2 = c^2 + 2ab\cos(\theta)$. So our chord length is computed as $c^2 = 1 + 1 - 2\cos(\theta) \rightarrow c = \sqrt{2 - 2\cos(\theta)}$. Since we have three classes of chords we can then compute their lengths as $\sqrt{2 - 2\cos(\frac{2\pi}{7})}$, $\sqrt{2 - 2\cos(\frac{4\pi}{7})}$, $\sqrt{2 - 2\cos(\frac{8\pi}{7})}$ respectively. Now we want the sum of the squares of the lengths so we really are trying to compute

$$7\left(\left(\sqrt{2 - 2\cos(\frac{2\pi}{7})}\right)^2 + \left(\sqrt{2 + 2\cos(\frac{4\pi}{7})}\right)^2 + \left(\sqrt{2 + 2\cos(\frac{8\pi}{7})}\right)^2\right) =$$

$$7\left(6 + 2\cos(\frac{2\pi}{7}) + 2\cos(\frac{4\pi}{7}) + 2\cos(\frac{8\pi}{7})\right)$$

Evaluating that last expression can be a little tricky. We can use Euler's Formula to expand cosine in terms of complex exponentials to yield

$$2\cos(\frac{2\pi}{7}) + 2\cos(\frac{4\pi}{7}) + 2\cos(\frac{8\pi}{7}) =$$

$$e^{\frac{2i\pi}{7}} + e^{\frac{-2i\pi}{7}} + e^{\frac{4i\pi}{7}} + e^{\frac{-4i\pi}{7}} + e^{\frac{8i\pi}{7}} + e^{\frac{-8i\pi}{7}}$$

Now we can put these into a standard form by recalling $e^{\frac{-2i\pi}{7}} = e^{\frac{12i\pi}{7}}$ (going counterclockwise a small amount the unit circle is the same as going clockwise a large amount). So we write this as:

$$e^{\frac{2i\pi}{7}} + e^{\frac{12i\pi}{7}} + e^{\frac{4i\pi}{7}} + e^{\frac{10i\pi}{7}} + e^{\frac{8i\pi}{7}} + e^{\frac{6i\pi}{7}}$$

Now we recall that sum of all the roots of unity satisfies

$$\sum_{k=0}^{n-1} e^{2ki\pi} n = 0$$

And we have 6 distinct 7th roots of unity here. So it must be true that:

$$1 + e^{\frac{2i\pi}{7}} + e^{\frac{12i\pi}{7}} + e^{\frac{4i\pi}{7}} + e^{\frac{10i\pi}{7}} + e^{\frac{8i\pi}{7}} + e^{\frac{6i\pi}{7}} = 0$$

So we conclude that:

$$e^{\frac{2i\pi}{7}} + e^{\frac{12i\pi}{7}} + e^{\frac{4i\pi}{7}} + e^{\frac{10i\pi}{7}} + e^{\frac{8i\pi}{7}} + e^{\frac{6i\pi}{7}} = -1$$

So our total squared sum is

$$7(6-(-1)) = 7*7 = 49$$

Giving us our answer.

Question 10. (Originally proposed by Terence Coelho, Ph.D Rutgers University) In the exotic foreign planet known as Philadelphia, PA, A day consists of 25 hours, an hour consists of 61 minutes, a minute consists of 61 seconds and a second consists of 1001 miniseconds. Where whenever an hour passes the minute hand makes a full revolution of the clock, when a minute passes the second hand makes a full revolution, when a second passes the minisecond hand makes a full revolution. Suppose you have a Philadelphia clock which has an hour hand, minute hand, second hand and millisecond hand that each move at a constant speed throughout the day according to the above rates. The clock has 25 hour labels all evenly spaced, so unlike a regular clock, the hour hand makes ONE revolution of this clock per day. When the day starts, all the hands are aligned facing up. Over the course of a day how many times will all 4 hands perfectly line up?

Solution: We begin by modelling our hands of a clock. Since the hands are uniformly moving in a circle it makes sense to use complex exponentials to model them. We say the first hand moves according to $e^{2\pi i t}$ where t is measured in days. So every day the hour hand makes a full revolution of the day. The next hand is the minute hand which moves according to $e^{2\pi i 25t}$. Interpreted literally this means the minute hand makes a full revolution whenever $\frac{1}{25}$ of a day has passed (i.e. a revolution occurs whenever an hour has passed). We next model the second hand as $e^{2\pi i 25*61t}$ this means the second hand makes a full revolution whenever $\frac{1}{25*61}$ of a day has passed (i.e. whenever a minute has passed). Now our minisecond hand gets modelled as $e^{2\pi i 25*61*61t}$ meaning that it makes a full revolution whenever a second has passed.As a reality check at $t = 0$ all of these equal 1 on the unit circle, so our condition of all lining up at the start of the day is met. So for all the hands to align means the hour, minute, second, millisecond function need to align: i.e.

$$e^{2\pi i t} = e^{2\pi i 25t} = e^{2\pi i 25*61t} = e^{2\pi i 25*61*61t}$$

Considered individually we have:

$$e^{2\pi i t} = e^{2\pi i 25t} \to e^{2\pi i 24t} = 1$$

$$e^{2\pi i 25t} = e^{2\pi i 25*61t} \to e^{2\pi i 25*60t} = 1$$

$$e^{2\pi i 25*61t} = e^{2\pi i 25*61*61t} \to e^{2\pi i 25*61*60t} = 1$$

Recall that t measures progress through the day so it ranges from 0 to 1 (0 inclusive and 1 non inclusive, 12:00AM this coming evening always counts as tomorrow!)

So $e^{2\pi i 24t} = 1$ can only happen if $t = \frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \ldots \frac{23}{24}$

Similarly: $\to e^{2\pi i 25*60t} = 1$ can only happen if t is an integer multiple of $\frac{1}{25*60}$. And lastly $e^{2\pi i 25*61*60t} = 1$ can only happen if t is an integer multiple of $\frac{1}{25*60*61}$. Now the last condition is extremely weak. It's implied by the previous condition. So we really have two conditions here:

t is an integer multiple of $\frac{1}{25*60}$ and t is an integer multiple of $\frac{1}{24}$. We can now compute the least common multiple of $25*60$ and 24 to try to combine the two conditions into one. We compute: $25*60 = 5^3*3*2^2$ and $24 = 2^3*3$ to find their least common multiple 5^3*3*2^3. And so we now simply require that t is an integer multiple of $\frac{2}{5^3*3*2^3}$ and t is an integer multiple of $\frac{5^3}{5^3*3*2^3}$. Finally we can conclude, we simply need to compute all the multiples of $2*5^3$ of the integers from 0 to 5^3*3*2^3-1. There are exactly $\lfloor\frac{5^3*3*2^3-1}{n}\rfloor+1$ such multiples divisible by n so we substitute:

$$\lfloor\frac{5^3*3*2^3-1}{2*5^3}\rfloor + 1 = 3*2^2+1 = 13$$

It's also worth pointing out that this is simply $GCD(24, 25 * 60) + 1$ for those a bit more arithmetically skilled.